

Title	REAL HYPERSURFACES WITH KILLING STRUCTURE JACOBI OPERATOR IN THE COMPLEX HYPERBOLIC QUADRIC
Author(s)	Suh, Young Jin
Citation	Osaka Journal of Mathematics. 58(1) p.1-p.28
Issue Date	2021-01
oaire:version	VoR
URL	<a href="https://doi.org/10.18910/78987">https://doi.org/10.18910/78987</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

# REAL HYPERSURFACES WITH KILLING STRUCTURE JACOBI OPERATOR IN THE COMPLEX HYPERBOLIC QUADRIC

YOUNG JIN SUH

(Received October 3, 2018, revised August 21, 2019)

## Abstract

First we introduce the notion of Killing structure Jacobi operator for real hypersurfaces in the complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^0/SO_2SO_m$ . Next we give a complete classification of real hypersurfaces in  $Q^{m*} = SO_{2,m}^0/SO_2SO_m$  with Killing structure Jacobi operator.

This work was supported by grant Proj. No. NRF-2018-R1D1A1B-05040381 from National Research Foundation of Korea

## 1. Introduction

In case of Hermitian symmetric space of rank 1, we say a complex projective space  $\mathbb{C}P^m$  and a complex hyperbolic space  $\mathbb{C}H^m$ . In the complex projective space  $\mathbb{C}P^m$ , a full classification of real hypersurfaces with isometric Reeb flow was obtained by Okumura in [16]. He proved that the Reeb flow on a real hypersurface in  $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$  is isometric if and only if  $M$  is an open part of a tube around a totally geodesic  $\mathbb{C}P^k \subset \mathbb{C}P^m$  for some  $k \in \{0, \dots, m-1\}$ . Moreover, Takagi [41] gave a complete classification of homogeneous hypersurfaces in  $\mathbb{C}P^m$  and Kimura and etc., [7] considered the notion GTW Reeb parallel shape operator. In the complex hyperbolic space  $\mathbb{C}H^m$ , Montiel and Romero [13] have given a complete classification of real hypersurface with isometric Reeb flow.

As another kind of Hermitian symmetric space with rank 2 of non-compact type different from the above ones, we can give the example of complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^0/SO_2SO_m$ . By using the method given in Kobayashi and Nomizu [12], Chapter XI, Example 10.6, the complex hyperbolic quadric  $Q^{m*} = SO_{2,m}^0/SO_2SO_m$  can be immersed in indefinite complex hyperbolic space  $CH_1^{m+1}$  as a space-like complex hypersurface (see Montiel and Romero [15] and Suh [34]). The complex hyperbolic quadric  $Q^{m*}$  is the non-compact Hermitian symmetric space  $SO_{2,m}^0/SO_2SO_m$  of rank 2 and also can be regarded as a kind of real Grassmann manifold of all oriented space-like 2-dimensional subspaces in indefinite flat Riemannian space  $\mathbb{R}_2^{m+2}$  (see Montiel and Romero [14] and [15]). Accordingly, the complex hyperbolic quadric admits both a complex conjugation structure  $A$  and a Kähler structure  $J$ , which anti-commutes with each other, that is,  $AJ = -JA$ . Then for  $m \geq 2$  the triple  $(Q^{m*}, J, g)$  is a Hermitian symmetric space of noncompact type with rank 2 and its minimal sectional curvature is equal to  $-4$  (see Klein [8] and Reckziegel [22]).

Now let us consider a real hypersurface in the complex hyperbolic quadric  $Q^{m*}$  with isometric Reeb flow. Then from the view of the previous results a natural expectation might be the totally geodesic  $Q^{m-1*} \subset Q^{m*}$ . But, suprisingly, in the complex hyperbolic quadric  $Q^{m*}$  the situation is quite different from the above ones. Recently, Suh [34] has introduced the following result:

**Theorem A.** *Let  $M$  be a real hypersurface of the complex hyperbolic quadric  $Q^{m*} = SO_{m,2}^o/SO_m SO_2$ ,  $m \geq 3$ . The Reeb flow on  $M$  is isometric if and only if  $m$  is even, say  $m = 2k$ , and  $M$  is locally congruent to an open part of a tube around a totally geodesic  $\mathbb{C}H^k \subset Q^{2k*}$  or a horosphere whose center at infinity is  $\mathfrak{A}$ -isotropic singular.*

Jacobi fields along geodesics of a given Riemannian manifold  $(M, g)$  satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if  $R$  denotes the curvature operator of  $M$ , and  $X$  is tangent vector field to  $M$ , then the Jacobi operator  $R_X \in \text{End}(T_x M)$  with respect to  $X$  at  $x \in M$ , defined by  $(R_X Y)(x) = (R(Y, X)X)(x)$  for any  $Y \in T_x M$ , becomes a self adjoint endomorphism of the tangent bundle  $TM$  of  $M$ . Thus, each tangent vector field  $X$  to  $M$  provides a Jacobi operator  $R_X$  with respect to  $X$ . In particular, for the Reeb vector field  $\xi$ , the Jacobi operator  $R_\xi$  is said to be a *structure Jacobi operator*.

Recently Ki, Pérez, Santos and Suh [5] have investigated the Reeb parallel structure Jacobi operator in the complex space form  $M_m(c)$ ,  $c \neq 0$  and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [20] have investigated real hypersurfaces  $M$  in  $G_2(\mathbb{C}^{m+2})$  with parallel structure Jacobi operator, that is,  $\nabla_X R_\xi = 0$  for any tangent vector field  $X$  on  $M$ . Jeong, Suh and Woo [4] and Pérez and Santos [18] have generalized such a notion to the recurrent structure Jacobi operator, that is,  $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$  for a certain 1-form  $\beta$  and any vector fields  $X, Y$  on  $M$  in  $G_2(\mathbb{C}^{m+2})$ . Moreover, Pérez, Santos and Suh [19] have further investigated the property of the Lie  $\xi$ -parallel structure Jacobi operator in complex projective space  $\mathbb{C}P^m$ , that is,  $\mathcal{L}_\xi R_\xi = 0$ .

The Reeb vector field  $\xi$  is *Killing* on  $M$  in  $Q^{m*}$  if and only if  $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$  for any vector fields  $X$  and  $Y$  on  $M$ . As a generalization of such a Killing vector field first Yano [42] defined the notion of *Killing tensor* as follows:

A skew symmetric tensor  $T_{i_1 \dots i_r}$  is called a *Killing tensor* of order  $r$  if it satisfies

$$\nabla_{i_1} T_{i_2 \dots i_{r+1}} + \nabla_{i_2} T_{i_1 \dots i_{r+1}} = 0.$$

Next Blair [2] has applied the notion of Killing tensor to a tensor field of  $T$  type  $(1, 1)$  on a Riemannian manifold and a geodesic  $\gamma$  on  $M$ . If we denote by  $\gamma'$  the tangent vector of the geodesic  $\gamma$ , then  $T\gamma'$  is parallel along the geodesic  $\gamma$  for the Killing tensor field  $T$ . Geometrically, this means that  $(\nabla_{\gamma'} T)\gamma' = 0$  along a geodesic  $\gamma$  on  $M$ . If this is the case for any geodesic on  $M$ , we have

$$(\nabla_X T)X = 0 \quad \text{or equivalently} \quad (\nabla_X T)Y + (\nabla_Y T)X = 0$$

for any vector fields  $X$  and  $Y$  on  $M$ . In this case we say that the tensor  $T$  is a *Killing tensor field of type  $(1, 1)$* .

Now we consider such a situation to the structure Jacobi operator  $R_\xi$ , which is a tensor field of type  $(1, 1)$  on a real hypersurface  $M$  in  $Q^{m*}$ . The structure Jacobi operator  $R_\xi$  of  $M$  in  $Q^m$  is said to be *Killing* if the structure Jacobi operator  $R_\xi$  satisfies

$$(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$$

for any  $X, Y \in T_z M$ ,  $z \in M$ . The equation is equivalent to  $(\nabla_X R_\xi)X = 0$  for any  $X \in T_z M$ ,  $z \in M$ , because of polarization. Moreover, we can give the geometric meaning of the Killing Jacobi operator as follows:

When we consider a geodesic  $\gamma$  with initial conditions such that  $\gamma(0) = z$  and  $\dot{\gamma}(0) = X$ . Then the transformed vector field  $R_\xi \dot{\gamma}$  is Levi-Civita *parallel* along the geodesic  $\gamma$  of the vector field  $X$  (see Blair [2] and Tachibana [40]).

In addition to the complex structure  $J$  there is another distinguished geometric structure on  $Q^{m*}$ , namely a parallel rank two vector bundle  $\mathfrak{A}$  which contains an  $S^1$ -bundle of real structures, that is, complex conjugations  $A$  on the tangent spaces of  $Q^{m*}$ . This geometric structure determines a maximal  $\mathfrak{A}$ -invariant subbundle  $\mathcal{Q}$  of the tangent bundle  $TM$  of a real hypersurface  $M$  in  $Q^{m*}$  as follows:

$$\mathcal{Q} = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}\}.$$

Recall that a nonzero tangent vector  $W \in T_{[z]}Q^{m*}$  is called singular if it is tangent to more than one maximal flat in  $Q^{m*}$ . There are two types of singular tangent vectors for the complex hyperbolic quadric  $Q^{m*}$ :

1. If there exists a conjugation  $A \in \mathfrak{A}$  such that  $W \in V(A)$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -principal.
2. If there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $W/\|W\| = (X + JY)/\sqrt{2}$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -isotropic

where  $V(A) = \{X \in T_{[z]}Q^{m*} \mid AX = X\}$  and  $JV(A) = \{X \in T_{[z]}Q^{m*} \mid AX = -X\}$ ,  $[z] \in Q^{m*}$ , are the  $(+1)$ -eigenspace and  $(-1)$ -eigenspace for the involution  $A$  on  $T_{[z]}Q^{m*}$ ,  $[z] \in Q^{m*}$ .

In the study of real hypersurfaces in the complex quadric  $Q^m$  we considered the notion of parallel Ricci tensor, that is,  $\nabla \text{Ric} = 0$  (see Suh [31]). But from the assumption of Ricci parallel, it was difficult for us to derive the fact that either the unit normal  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal. So in [31] we gave a classification with the further assumption of  $\mathfrak{A}$ -isotropic. But fortunately, if we consider a Hopf real hypersurfaces, which is defined by  $S\xi = \alpha\xi$  for the Reeb function  $\alpha = g(S\xi, \xi)$  and the shape operator  $S$ , in the complex hyperbolic quadric  $Q^{m*}$  with Killing structure Jacobi operator, we can assert that the unit normal vector field  $N$  becomes either  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal as follows:

**Main Theorem 1.** *Let  $M$  be a Hopf real hypersurface in  $Q^{m*}$ ,  $m \geq 3$ , with Killing structure Jacobi operator. Then the unit normal vector field  $N$  is singular; that is,  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal.*

When we consider a hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$ , the unit normal vector field  $N$  of  $M$  in  $Q^{m*}$  can be divided into two cases :  $N$  is  $\mathfrak{A}$ -isotropic or

$\mathfrak{A}$ -principal (see [34], [35] and [27]). In the first case where  $M$  has an  $\mathfrak{A}$ -isotropic unit normal  $N$ , we have asserted in [34] and [35] that  $M$  is locally congruent to a tube over a totally geodesic complex hyperbolic space  $\mathbb{C}H^k$  in  $Q^{2k*}$  or a horosphere with  $\mathfrak{A}$ -isotropic unit normal vector field centered at the infinity. In the second case when  $N$  is  $\mathfrak{A}$ -principal we have proved that  $M$  is locally congruent to a tube over a totally geodesic and totally real submanifold  $Q^{m-1*}$  in  $Q^{m*}$  (see [34], [36] and [38]).

In this paper we consider the case that the structure Jacobi operator  $R_\xi$  of  $M$  in  $Q^{m*}$  is Killing, that is,  $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$  for any tangent vector field  $X$  and  $Y$  on  $M$ , and we prove the following

**Main Theorem 2.** *There does not exist a Hopf hypersurface in  $Q^{m*}$ ,  $m \geq 3$  with Killing structure Jacobi operator and  $\mathfrak{A}$ -principal unit normal vector field.*

Now it remains to prove the case that the unit normal vector field is  $\mathfrak{A}$ -isotropic. Then by our Main Theorems 1 and 2, we give a classification of real hypersurfaces in  $Q^{m*}$  with Killing structure Jacobi operator as follows:

**Main Theorem 3.** *Let  $M$  be a Hopf hypersurface in  $Q^{m*}$ ,  $m \geq 3$  with Killing structure Jacobi operator. If the Reeb function is constant along the Reeb direction, then  $M$  has 4 distinct constant principal curvatures*

$$\alpha, \quad \beta = 0, \quad \lambda_1 \quad \lambda_2.$$

Here the corresponding eigen spaces  $\xi \in T_\alpha$ ,  $T_\beta = Q^\perp$ , and  $T_{\lambda_1} \oplus T_{\lambda_2} = Q$ , where the principal curvatures  $\lambda_1$  and  $\lambda_2$  are two distinct constants given by

$$\lambda_1 = \frac{\alpha(\alpha^2 - 1) + \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}$$

and

$$\lambda_2 = \frac{\alpha(\alpha^2 - 1) - \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}.$$

with multiplicities  $(m - 2)$  respectively and  $\alpha^2 > 2\sqrt{2} + 1$ .

**REMARK 1.1.** In [29] Suh has proved that the Reeb function  $\alpha = g(S\xi, \xi)$  is constant for real hypersurfaces with singular normal vector field in the complex quadric  $Q^m$ . But in the complex hyperbolic quadric  $Q^{m*}$  the Reeb function  $\alpha$  is constant only if the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal (see Suh, Pérez and Woo [39]). Until now it does not known to us whether the Reeb function  $\alpha$  is constant for real hypersurfaces in the complex hyperbolic quadric  $Q^{m*}$  with  $\mathfrak{A}$ -isotropic unit normal vector field.

The subbundle  $Q$  mentioned in Main Theorem 3 is the maximal invariant subbundle of  $T_z M$ ,  $z \in M$ , such that  $Q \oplus Q^\perp = [\xi]^\perp$ , where  $Q^\perp = \text{Span}\{A\xi, AN\}$  and  $[\xi]^\perp$  denotes the orthogonal complement of the Reeb vector field  $\xi$  in  $T_z M$ ,  $z \in M$ , in  $Q^{m*}$ .

When we consider a parallel structure Jacobi operator on  $M$  in  $Q^{m*}$ , we know that  $(\nabla_X R_\xi)Y = 0$  for any vector fields  $X$  and  $Y$  on  $M$ . This gives a condition stronger than

the notion of Killing structure Jacobi operator. So naturally it satisfies the assumptions of Killing in Main Theorems 1, 2 and 3. For the case of isotropic unit normal  $N$ , it can be easily checked that the results in our Main Theorem 3 do not satisfy the strong assumption of parallel structure Jacobi operator. So we also conclude the following

**Corollary** (see [39]). *There does not exist a Hopf hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ , with parallel structure Jacobi operator.*

## 2. The complex hyperbolic quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric  $Q^{m*}$  different from the complex quadric  $Q^m$ . This section is due to Klein and Suh [10].

The  $m$ -dimensional complex hyperbolic quadric  $Q^{m*}$  is the non-compact dual of the  $m$ -dimensional complex quadric  $Q^m$ , which is a kind of Hermitian symmetric space of non-compact type with rank 2 (see Besse [1], and Helgason [3]).

The complex hyperbolic quadric  $Q^{m*}$  cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space  $\mathbb{C}H^{m+1}$ . In fact, Smyth [24, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in  $\mathbb{C}H^{m+1}$  is totally geodesic. This is in marked contrast to the situation for the complex quadric  $Q^m$ , which can be realized as a homogeneous complex hypersurface of the complex projective space  $\mathbb{C}P^{m+1}$  in such a way that the shape operator for any unit normal vector to  $Q^m$  is a real structure on the corresponding tangent space of  $Q^m$ , see [8] and [22]. Another related result by Smyth, [24, Theorem 1], which states that any complex hypersurface  $\mathbb{C}H^{m+1}$  for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of  $Q^{m*}$  as a complex hypersurface of  $\mathbb{C}H^{m+1}$  with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric  $Q^{m*}$  as the quotient manifold  $SO_{2,m}^0/SO_2SO_m$ . As  $Q^{1*}$  is isomorphic to the real hyperbolic space  $\mathbb{R}H^2 = SO_{1,2}^0/SO_2$ , and  $Q^{2*}$  is isomorphic to the Hermitian product of complex hyperbolic spaces  $\mathbb{C}H^1 \times \mathbb{C}H^1$ , we suppose  $m \geq 3$  in the sequel and throughout this paper. Let  $G := SO_{2,m}^0$  be the transvection group of  $Q^{m*}$  and  $K := SO_2SO_m$  be the isotropy group of  $Q^{m*}$  at the “origin”  $p_0 := eK \in Q^{m*}$ . Then

$$\sigma : G \rightarrow G, g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of  $G$  with  $\text{Fix}(\sigma)_0 = K$ , and therefore  $Q^{m*} = G/K$  is a Riemannian symmetric space. The center of the isotropy group  $K$  is isomorphic to  $SO_2$ , and therefore  $Q^{m*}$  is in fact a Hermitian symmetric space.

The Lie algebra  $\mathfrak{g} := \mathfrak{so}_{2,m}$  of  $G$  is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m+2, \mathbb{R}) \mid X^t \cdot s = -s \cdot X\}$$

(see [11, p. 59]). In the sequel we will write members of  $\mathfrak{g}$  as block matrices with respect to the decomposition  $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$ , i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where  $X_{11}, X_{12}, X_{21}, X_{22}$  are real matrices of the dimension  $2 \times 2, 2 \times m, m \times 2$  and  $m \times m$ , respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{12}^t = X_{21}, X_{22}^t = -X_{22} \right\}.$$

The linearisation  $\sigma_L = \text{Ad}(s) : \mathfrak{g} \rightarrow \mathfrak{g}$  of the involutive Lie group automorphism  $\sigma$  induces the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ , where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \text{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} \mid sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group  $K$ , and the  $2m$ -dimensional linear subspace

$$\mathfrak{m} = \text{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} \mid sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \mid X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space  $T_{p_0}Q^{m*}$ . Under the identification  $T_{p_0}Q^{m*} \cong \mathfrak{m}$ , the Riemannian metric  $g$  of  $Q^{m*}$  (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2} \text{tr}(Y^t \cdot X) = \text{tr}(Y_{12} \cdot X_{21}) \quad \text{for } X, Y \in \mathfrak{m}.$$

$g$  is clearly  $\text{Ad}(K)$ -invariant, and therefore corresponds to an  $\text{Ad}(G)$ -invariant Riemannian metric on  $Q^{m*}$ . The complex structure  $J$  of the Hermitian symmetric space is given by

$$JX = \text{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} 0 & 1 \\ -1 & 0 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \in K.$$

Because  $j$  is in the center of  $K$ , the orthogonal linear map  $J$  is  $\text{Ad}(K)$ -invariant, and thus defines an  $\text{Ad}(G)$ -invariant Hermitian structure on  $Q^{m*}$ . By identifying the multiplication with the unit complex number  $i$  with the application of the linear map  $J$ , the tangent spaces of  $Q^{m*}$  thus become  $m$ -dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As mentioned for the complex quadric (again compare [8], [9], and [22]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an  $S^1$ -bundle  $\mathfrak{A}$  of real structures. The situation here differs from that of the complex quadric in that for  $Q^{m*}$ , the real structures in  $\mathfrak{A}$  cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show,  $\mathfrak{A}$  still plays an important role in the description of the geometry of  $Q^{m*}$ .



Let

$$a_0 := \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}.$$

Note that we have  $a_0 \notin K$ , but only  $a_0 \in O_2 SO_m$ . However,  $\text{Ad}(a_0)$  still leaves  $\mathfrak{m}$  invariant, and therefore defines an  $\mathbb{R}$ -linear map  $A_0$  on the tangent space  $\mathfrak{m} \cong T_{p_0} Q^{m*}$ .  $A_0$  turns out to be an involutive orthogonal map with  $A_0 \circ J = -J \circ A_0$  (i.e.  $A_0$  is anti-linear with respect to the complex structure of  $T_{p_0} Q^{m*}$ ), and hence a real structure on  $T_{p_0} Q^{m*}$ . But  $A_0$  commutes with  $\text{Ad}(g)$  not for all  $g \in K$ , but only for  $g \in SO_m \subset K$ . More specifically, for  $g = (g_1, g_2) \in K$  with  $g_1 \in SO_2$  and  $g_2 \in SO_m$ , say  $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$  with  $t \in \mathbb{R}$  (so that  $\text{Ad}(g_1)$  corresponds to multiplication with the complex number  $\mu := e^{it}$ ), we have

$$A_0 \circ \text{Ad}(g) = \mu^{-2} \cdot \text{Ad}(g) \circ A_0.$$

This equation shows that the object which is  $\text{Ad}(K)$ -invariant and therefore geometrically relevant is not the real structure  $A_0$  by itself, but rather the “circle of real structures”

$$\mathfrak{A}_{p_0} := \{\lambda A_0 | \lambda \in S^1\}.$$

$\mathfrak{A}_{p_0}$  is  $\text{Ad}(K)$ -invariant, and therefore generates an  $\text{Ad}(G)$ -invariant  $S^1$ -subbundle  $\mathfrak{A}$  of the endomorphism bundle  $\text{End}(TQ^{m*})$ , consisting of real structures on the tangent spaces of  $Q^{m*}$ . For any  $A \in \mathfrak{A}$ , the tangent line to the fibre of  $\mathfrak{A}$  through  $A$  is spanned by  $JA$ .

For any  $p \in Q^{m*}$  and  $A \in \mathfrak{A}_p$ , the real structure  $A$  induces a splitting

$$T_p Q^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space  $T_p Q^{m*}$ . Here  $V(A)$  resp.  $JV(A)$  are the  $(+1)$ -eigenspace resp. the  $(-1)$ -eigenspace of  $A$ . For every unit vector  $W \in T_p Q^{m*}$  there exist  $t \in [0, \frac{\pi}{4}]$ ,  $A \in \mathfrak{A}_p$  and orthonormal vectors  $X, Y \in V(A)$  so that

$$W = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [22, Proposition 3]. Here  $t$  is uniquely determined by  $W$ . The vector  $W$  is singular, i.e. contained in more than one Cartan subalgebra of  $\mathfrak{m}$ , if and only if either  $t = 0$  or  $t = \frac{\pi}{4}$  holds. The vectors with  $t = 0$  are called  $\mathfrak{A}$ -*principal*, whereas the vectors with  $t = \frac{\pi}{4}$  are called  $\mathfrak{A}$ -*isotropic*. If  $W$  is regular, i.e.  $0 < t < \frac{\pi}{4}$  holds, then also  $A$  and  $X, Y$  are uniquely determined by  $W$ .

The singular tangent vectors correspond to the values  $t = 0$  and  $t = \pi/4$ . If  $0 < t < \pi/4$  then the unique maximal flat containing  $W$  is  $\mathbb{R}X \oplus \mathbb{R}JY$ . Later we will need the eigenvalues and eigenspaces of the Jacobi operator  $R_W = R(\cdot, W)W$  for a singular unit tangent vector  $W$ .

1. If  $W$  is an  $\mathfrak{A}$ -principal singular unit tangent vector with respect to  $A \in \mathfrak{A}$ , then the eigenvalues of  $R_W$  are 0 and 2 and the corresponding eigenspaces are  $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$  and  $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$ , respectively.
2. If  $W$  is an  $\mathfrak{A}$ -isotropic singular unit tangent vector with respect to  $A \in \mathfrak{A}$  and  $X, Y \in V(A)$ , then the eigenvalues of  $R_W$  are 0, 1 and 4 and the corresponding eigenspaces are  $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$ ,  $T_z Q^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$  and  $\mathbb{R}JW$ , respectively.



Like for the complex quadric, the Riemannian curvature tensor  $\bar{R}$  of  $Q^{m*}$  can be fully described in terms of the “fundamental geometric structures”  $g$ ,  $J$  and  $\mathfrak{A}$ . In fact, under the correspondence  $T_{p_0}Q^{m*} \cong \mathfrak{m}$ , the curvature  $\bar{R}(X, Y)Z$  corresponds to  $-[[X, Y], Z]$  for  $X, Y, Z \in \mathfrak{m}$ , see [12, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}\bar{R}(X, Y)Z &= -g(Y, Z)X + g(X, Z)Y \\ &\quad - g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ &\quad - g(AY, Z)AX + g(AX, Z)AY \\ &\quad - g(JAY, Z)JAX + g(JAX, Z)JAY\end{aligned}$$

for arbitrary  $A \in \mathfrak{A}_{p_0}$ . Therefore the curvature of  $Q^{m*}$  is the negative of that of the complex quadric  $Q^m$ , compare [22, Theorem 1]. This confirms that the symmetric space  $Q^{m*}$  which we have constructed here is indeed the non-compact dual of the complex quadric.

### 3. Some general equations

Let  $M$  be a real hypersurface in the complex hyperbolic quadric  $Q^{m*}$  and denote by  $(\phi, \xi, \eta, g)$  the induced almost contact metric structure. Note that  $\xi = -JN$ , where  $N$  is a (local) unit normal vector field of  $M$ . The tangent bundle  $TM$  of  $M$  splits orthogonally into  $TM = C \oplus \mathbb{R}\xi$ , where  $C = \ker(\eta)$  is the maximal complex subbundle of  $TM$ . The structure tensor field  $\phi$  restricted to  $C$  coincides with the complex structure  $J$  restricted to  $C$ , and  $\phi\xi = 0$ .

At each point  $z \in M$  we define the maximal  $\mathfrak{A}$ -invariant subspace of  $T_zM$ ,  $z \in M$  as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

**Lemma 3.1** (see [29]). *For each  $z \in M$  we have*

- (i) *If  $N_z$  is  $\mathfrak{A}$ -principal, then  $\mathcal{Q}_z = C_z$ .*
- (ii) *If  $N_z$  is not  $\mathfrak{A}$ -principal, there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $N_z = \cos(t)X + \sin(t)JY$  for some  $t \in (0, \pi/4]$ . Then we have  $\mathcal{Q}_z = C_z \ominus \mathbb{C}(JX + Y)$ .*

We now assume that  $M$  is a Hopf hypersurface. Then for the Reeb vector field  $\xi$  the shape operator  $S$  becomes

$$S\xi = \alpha\xi$$

with the smooth function  $\alpha = g(S\xi, \xi)$  on  $M$ . When we consider a transform  $JX$  of the Kaehler structure  $J$  on the complex hyperbolic quadric  $Q^{m*}$  for any vector field  $X$  on  $M$  in  $Q^{m*}$ , we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal  $N$  to  $M$ .

Then we now consider the Codazzi equation

$$\begin{aligned}
 (3.1) \quad g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\
 &\quad - g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) \\
 &\quad - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z).
 \end{aligned}$$

Putting  $Z = \xi$  we get

$$\begin{aligned}
 g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= 2g(\phi X, Y) \\
 &\quad - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
 &\quad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
 &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\
 &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi S X, Y).
 \end{aligned}$$

Comparing the previous two equations and putting  $X = \xi$  yields

$$(3.2) \quad Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned}
 &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
 &= 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
 &\quad - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \\
 &\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi S X, Y).
 \end{aligned}$$

Altogether this implies

$$\begin{aligned}
 0 &= 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\
 &\quad - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
 &\quad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\
 &\quad - 2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
 &\quad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X).
 \end{aligned}$$

At each point  $z \in M$  we can choose  $A \in \mathfrak{A}_z$  such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 \leq t \leq \frac{\pi}{4}$  (see Proposition 3 in [22]). Note that  $t$  is a function on  $M$ . First of all, since  $\xi = -JN$ , we have

$$\begin{aligned}
 AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\
 \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\
 A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
 \end{aligned}$$

This implies  $g(\xi, AN) = 0$  and hence

$$0 = 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y)$$

$$\begin{aligned}
& -g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
& + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\
& + 2g(X, AN)g(\xi, A\xi)\eta(Y) - 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}$$

We have  $JA\xi = -AJ\xi = -AN$ , and inserting this into the previous equation implies

**Lemma 3.2.** *Let  $M$  be a Hopf hypersurface in the complex hyperbolic quadric  $Q^{m*}$  with (local) unit normal vector field  $N$ . For each point  $z \in M$  we choose  $A \in \mathfrak{A}_z$  such that  $N_z = \cos(t)Z_1 + \sin(t)JZ_2$  holds for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 \leq t \leq \frac{\pi}{4}$ . Then*

$$\begin{aligned}
0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\
&\quad - 2g(X, AN)g(Y, A\xi) + 2g(Y, AN)g(X, A\xi) \\
&\quad - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\}
\end{aligned}$$

holds for all vector fields  $X$  and  $Y$  on  $M$ .

We can write for any vector field  $Y$  on  $M$  in  $Q^{m*}$

$$AY = BY + \rho(Y)N,$$

where  $BY$  denotes the tangential component of  $AY$  and  $\rho(Y) = g(AY, N)$ .

If  $N$  is  $\mathfrak{A}$ -principal, that is,  $AN = N$ , we have  $\rho = 0$ , because  $\rho(Y) = g(Y, AN) = g(Y, N) = 0$  for any tangent vector field  $Y$  on  $M$  in  $Q^{m*}$ . So we have  $AY = BY$  for any tangent vector field  $Y$  on  $M$  in  $Q^{m*}$ . Otherwise we can use Lemma 3.1 to calculate  $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$  for any tangent vector field  $Y$  on  $M$  in  $Q^{m*}$ . From this, together with Lemma 3.2, we have proved

**Lemma 3.3.** *Let  $M$  be a Hopf hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi - \beta\xi) + 2g(X, B\xi - \beta\xi)\phi B\xi,$$

where the function  $\beta$  is given by  $\beta = g(\xi, A\xi) = -g(N, AN)$ .

If the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal, we can choose a real structure  $A \in \mathfrak{A}$  such that  $AN = N$ . Then we have  $\rho = 0$  and  $\phi B\xi = -\phi\xi = 0$ , and therefore

$$(3.3) \quad 2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

If  $N$  is not  $\mathfrak{A}$ -principal, we can choose a real structure  $A \in \mathfrak{A}$  as in Lemma 3.1 and get

$$\begin{aligned}
(3.4) \quad & \rho(X)(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi B\xi \\
&= -g(X, \phi(B\xi - \beta\xi))(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi(B\xi - \beta\xi) \\
&= \|B\xi - \beta\xi\|^2 \{g(X, U)\phi U - g(X, \phi U)U\} \\
&= \sin^2(2t) \{g(X, U)\phi U - g(X, \phi U)U\},
\end{aligned}$$

which is equal to 0 on  $\mathcal{Q}$  and equal to  $\sin^2(2t)\phi X$  on  $C \ominus \mathcal{Q}$ . Altogether we have proved:

**Lemma 3.4.** *Let  $M$  be a Hopf hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

*leaves  $\mathcal{Q}$  and  $C \ominus \mathcal{Q}$  invariant and we have*

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

*and*

$$2S\phi S - \alpha(\phi S + S\phi) = -2\beta^2\phi \text{ on } C \ominus \mathcal{Q},$$

*where  $\beta = g(A\xi, \xi) = -\cos 2t$  as in section 3.*

Then from the equation of Gauss the curvature tensor  $R$  of  $M$  in complex quadric  $Q^{m*}$  is defined so that

$$\begin{aligned} R(X, Y)Z &= -g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z \\ &\quad -g(AY, Z)(AX)^T + g(AX, Z)(AY)^T - g(JAY, Z)(JAX)^T \\ &\quad + g(JAX, Z)(JAY)^T + g(SY, Z)SX - g(SX, Z)SY, \end{aligned}$$

where  $(AX)^T$  and  $S$  denote the tangential component of the vector field  $AX$  and the shape operator of  $M$  in  $Q^{m*}$  respectively.

From this, putting  $Y = Z = \xi$  and using  $g(A\xi, N) = 0$ , the structure Jacobi operator is defined by

$$\begin{aligned} R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi - g(A\xi, \xi)(AX)^T + g(AX, \xi)A\xi \\ &\quad + g(X, AN)(AN)^T + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned}$$

Then we may put the following

$$(AY)^T = AY - g(AY, N)N.$$

Now let us denote by  $\nabla$  and  $\bar{\nabla}$  the covariant derivative of  $M$  and the covariant derivative of  $Q^{m*}$  respectively. Then by using the Gauss and Weingarten formulas we can assert the following

**Lemma 3.5.** *Let  $M$  be a real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ . Then*

$$\begin{aligned} (3.5) \quad \nabla_X(AY)^T &= q(X)JAY + A\nabla_X Y + g(SX, Y)AN \\ &\quad - g(\{q(X)JAY + A\nabla_X Y + g(SX, Y)AN\}, N)N \\ &\quad + g(AY, N)SX. \end{aligned}$$

Proof. First let us use the Gauss formula to  $(AY)^T = AY - g(AY, N)N$ . Then it follows that

$$\begin{aligned}\nabla_X(AY)^T &= \bar{\nabla}_X(AY)^T - \sigma(X, (AY)^T) \\ &= \bar{\nabla}_X\{AY - g(AY, N)N\} - g(SX, (AY)^T)N \\ &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - g((\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y, N)N - g(AY, \bar{\nabla}_X N)N \\ &\quad - g(AY, N)\bar{\nabla}_X N - g(SX, (AY)^T)N,\end{aligned}$$

where  $\sigma$  denotes the second fundamental form and  $N$  the unit normal vector field on  $M$  in  $Q^{m*}$ . Then from this, if we use Weingarten formula  $\bar{\nabla}_X N = -SX$ , then we get the above formula.  $\square$

By putting  $Y = \xi$  and using  $g(A\xi, N) = 0$ , we have

$$(3.6) \quad \begin{aligned}\nabla_X(A\xi) &= q(X)JA\xi + A\phi SX + \alpha\eta(X)AN \\ &\quad - \{q(X)g(JA\xi, N) + g(A\phi SX, N) + \alpha\eta(X)g(AN, N)\}N.\end{aligned}$$

Moreover, let us also use Gauss and Weingarten formula to  $(AN)^T = AN - g(AN, N)N$ . Then it follows that

$$\begin{aligned}(3.7) \quad \nabla_X(AN)^T &= \bar{\nabla}_X(AN)^T - \sigma(X, (AN)^T) \\ &= \bar{\nabla}_X\{AN - g(AN, N)N\} - \sigma(X, (AN)^T) \\ &= (\bar{\nabla}_X A)N + A\bar{\nabla}_X N - g((\bar{\nabla}_X A)N + A\bar{\nabla}_X N, N) \\ &\quad - g(AN, \bar{\nabla}_X N)N - g(AN, N)\bar{\nabla}_X N - \sigma(X, (AN)^T) \\ &= q(X)JAN - ASX - g(q(X)JAN - ASX, N)N + g(AN, N)SX.\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}(3.8) \quad X\beta &= X(g(A\xi, \xi)) \\ &= g((\bar{\nabla}_X A)\xi + A\bar{\nabla}_X \xi, \xi) + g(A\xi, \bar{\nabla}_X \xi) \\ &= g(q(X)JA\xi + A\phi SX + g(SX, \xi)AN, \xi) + g(A\xi, \phi SX + g(SX, \xi)N) \\ &= 2g(A\phi SX, \xi).\end{aligned}$$

#### 4. Some Important Lemmas and Proof of Theorem 1

The curvature tensor  $R(X, Y)Z$  for a Hopf real hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$  induced from the curvature tensor of  $Q^{m*}$  is given in section 3. Now the structure Jacobi operator  $R_\xi$  can be rewritten as follows:

$$\begin{aligned}(4.1) \quad R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi - \beta(AX)^T + g(AX, \xi)A\xi + g(AX, N)(AN)^T \\ &\quad + \alpha SX - g(SX, \xi)S\xi,\end{aligned}$$

where we have put  $\alpha = g(S\xi, \xi)$  and  $\beta = g(A\xi, \xi)$ , because we assume that  $M$  is Hopf. The Reeb vector field  $\xi = -JN$  and the anti-commuting property  $AJ = -JA$  gives that the function  $\beta$  becomes  $\beta = -g(AN, N)$ . When this function  $\beta = g(A\xi, \xi)$  identically vanishes,

we say that a real hypersurface  $M$  in  $Q^{m*}$  is  $\mathfrak{A}$ -isotropic as in section 1.

Here let us differentiate the structure Jacobi operator  $R_\xi$  along any direction  $X$  on  $M$  in the complex hyperbolic quadric  $Q^{m*}$ . Then (4.1), together with (3.5), (3.6), (3.7), give that

$$\begin{aligned}
 (4.2) \quad \nabla_X R_\xi(Y) &= \nabla_X(R_\xi(Y)) - R_\xi(\nabla_X Y) \\
 &= g(\phi S X, Y)\xi + \eta(Y)\phi S X - (X\beta)(AY)^T \\
 &\quad - \beta[q(X)JAY + A\nabla_X Y + g(SX, Y)AN \\
 &\quad - g(\{q(X)JAY + A\nabla_X Y + g(SX, Y)AN\}, N)N \\
 &\quad + g(AY, N)SX] \\
 &\quad + g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN, Y)A\xi \\
 &\quad + g(AY, \xi)[g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN \\
 &\quad - \{q(X)g(JA\xi, N) + g(A\phi S X, N) + \alpha\eta(X)g(AN, N)\}N] \\
 &\quad + [g(q(X)JAN - ASX + g(AN, N)SX, Y)(AN)^T \\
 &\quad + g(Y, (AN)^T)\{q(X)JAN - ASX + g(AN, N)SX \\
 &\quad - g(q(X)JAN - ASX, N)N\}] \\
 &\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - X(\alpha^2)\eta(Y)\xi \\
 &\quad - \alpha^2(\nabla_X \eta)(Y)\xi - \alpha^2\eta(Y)\nabla_X \xi,
 \end{aligned}$$

where we have used  $g(A\xi, N) = 0$ , and  $N$  the unit normal to  $M$  in  $Q^{m*}$ .

Here let us assume that the structure Jacobi operator is Killing, that is,  $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$  for any tangent vector fields  $X$  and  $Y$  on  $M$  in  $Q^{m*}$ . Then from this, together with (4.1), we have the following

$$\begin{aligned}
 (4.3) \quad 0 &= \nabla_X R_\xi(Y) + \nabla_Y R_\xi(X) \\
 &= \{g(\phi S X, Y) + g(\phi S Y, X)\}\xi + \eta(Y)\phi S X + \eta(X)\phi S Y \\
 &\quad - (X\beta)(AY)^T - (Y\beta)(AX)^T \\
 &\quad - \beta[q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(SX, Y)AN \\
 &\quad - g(\{q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(SX, Y)AN\}, N)N \\
 &\quad + g(AY, N)SX + g(AX, N)SY] \\
 &\quad + [g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN, Y) \\
 &\quad + g(q(Y)JA\xi + A\phi S Y + \alpha\eta(Y)AN, X)]A\xi \\
 &\quad + g(AY, \xi)[q(X)JA\xi + A\phi S X + \alpha\eta(X)AN \\
 &\quad - \{q(X)g(JA\xi, N) + g(A\phi S X, N) + \alpha\eta(X)g(AN, N)\}N] \\
 &\quad + g(AX, \xi)[q(Y)JA\xi + A\phi S Y + \alpha\eta(Y)AN \\
 &\quad - \{q(Y)g(JA\xi, N) + g(A\phi S Y, N) + \alpha\eta(Y)g(AN, N)\}N] \\
 &\quad + [g(q(X)JAN - ASX + g(AN, N)SX, Y)
 \end{aligned}$$

$$\begin{aligned}
& + g(q(Y)JAN - ASY + g(AN, N)SY, X)\{(AN)^T \\
& + g(Y, (AN)^T)\{q(X)JAN - ASX - g(q(X)JAN - ASX, N)N \\
& + g(AN, N)SX\} \\
& + g(X, (AN)^T)\{q(Y)JAN - ASY - g(q(Y)JAN - ASY, N)N \\
& + g(AN, N)SY\}] \\
& + (X\alpha)SY + (Y\alpha)SX + \alpha\{(\nabla_X S)Y + (\nabla_Y S)X\} \\
& - X(\alpha^2)\eta(Y)\xi - (Y\alpha^2)\eta(X)\xi - \alpha^2\{(\nabla_X \eta)(Y)\xi + (\nabla_Y \eta)(X)\xi\} \\
& - \alpha^2\{\eta(Y)\nabla_X \xi + \eta(X)\nabla_Y \xi\}.
\end{aligned}$$

From this, by taking the inner product of (4.3) with the Reeb vector field  $\xi$ , we have

$$\begin{aligned}
0 = & g((\phi S - S\phi)X, Y) - (X\beta)g(AY, \xi) - (Y\beta)g(AX, \xi) \\
& - \beta\{q(X)g(JAY, \xi) + q(Y)g(JAX, \xi) + g(A(\nabla_X Y + \nabla_Y X), \xi) \\
& + g(AY, N)g(SX, \xi) + g(AX, N)g(SY, \xi)\} \\
& + \{g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y) \\
& + g(q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN, X)\}g(A\xi, \xi) \\
& + g(AY, \xi)g(A\phi SX, \xi) + g(AX, \xi)g(A\phi SY, \xi) \\
& + g(Y, (AN)^T)\{g(q(X)JAN, \xi) - g(ASX, \xi) + g(AN, N)g(SX, \xi)\} \\
& + g(X, (AN)^T)\{g(q(Y)JAN, \xi) - g(ASY, \xi) + g(AN, N)g(SY, \xi)\} \\
& + \alpha(X\alpha)\eta(Y) + \alpha(Y\alpha)\eta(X) \\
& + \alpha\{g((\nabla_X S)Y, \xi) + g((\nabla_Y S)X, \xi)\} \\
& - X(\alpha^2)\eta(Y) - Y(\alpha^2)\eta(X) - \alpha^2(\nabla_X \eta)(Y) - \alpha^2(\nabla_Y \eta)(X).
\end{aligned}$$

Then, first, by putting  $Y = \xi$  and using  $g(A\xi, N) = 0$ , we have

$$\begin{aligned}
(4.4) \quad 0 = & - (X\beta)g(A\xi, \xi) - \beta g(A\phi SX, \xi) + \beta g(A\phi SX, \xi) + \beta g(A\phi SX, \xi) \\
& - (\xi\beta)g(AX, \xi) - \beta\{q(\xi)g(JAX, \xi) + g(A\nabla_\xi X, \xi) + \alpha g(AX, N)\} \\
& + \{g(q(\xi)JA\xi + A\phi S\xi + \alpha AN, X)\}g(A\xi, \xi) \\
& + g(X, AN)(q(\xi) - 2\alpha)\beta \\
= & - \beta\{g(A\phi SX, \xi) + g(A\nabla_\xi X, \xi) - (q(\xi) - 2\alpha)g(X, AN)\}.
\end{aligned}$$

Here if the function  $\beta = g(A\xi, \xi) = -\cos 2t = 0$ , we have  $t = \frac{\pi}{4}$ . Then the unit normal vector field  $N$  becomes

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for  $Z_1, Z_2 \in V(A)$  as in section 3, that is, the unit normal  $N$  is  $\mathfrak{U}$ -isotropic .

Now hereafter, from (4.4) let us consider the following case

$$(4.5) \quad 0 = \{g(A\phi SX, \xi) + g(A\nabla_\xi X, \xi) - (q(\xi) - 2\alpha)g(X, AN)\}.$$

On the other hand, by using (3.1) for any tangent vector field  $X \perp A\xi$ , we have



$$\begin{aligned}
 (4.6) \quad g(A\nabla_\xi X, \xi) &= g(\nabla_\xi X, A\xi) = -g(X, \nabla_\xi(A\xi)) \\
 &= -g(q(\xi)JA\xi + \alpha AN, X) = (q(\xi) - \alpha)g(AN, X).
 \end{aligned}$$

Then from (4.5) and (4.6) we have the following for any tangent vector field  $X$  orthogonal to  $A\xi$

$$\begin{aligned}
 (4.7) \quad 0 &= g(A\phi S X, \xi) + (q(\xi) - \alpha)g(AN, X) - (q(\xi) - 2\alpha)g(AN, X) \\
 &= g(A\phi S X, \xi) + \alpha g(AN, X) \\
 &= g(SAN + \alpha AN, X).
 \end{aligned}$$

So it follows that

$$(4.8) \quad g(S(AN)^T, (AN)^T) = -\alpha(1 - \beta^2),$$

where  $g((AN)^T, (AN)^T) = g(AN - g(AN, N)N, AN - g(AN, N)N) = 1 - g(AN, N)^2 = 1 - \beta^2$ .

On the other hand, by using (3.3) to the second term of (4.5) for  $X = (AN)^T$ , we have

$$\begin{aligned}
 (4.9) \quad g(A\nabla_\xi(AN)^T, \xi) &= g(q(\xi)\xi - S\xi + \alpha g(AN, N)A\xi, \xi) \\
 &= q(\xi) - \alpha - \alpha\beta^2,
 \end{aligned}$$

where we have used  $A^2 = I$  and  $g(AN, N) = -g(A\xi, \xi) = -\beta$ .

Then by putting  $X = (AN)^T$  in (4.5) and using (4.8) and (4.9), we have

$$\begin{aligned}
 (4.10) \quad 0 &= g(A\phi S(AN)^T, \xi) + g(A\nabla_\xi(AN)^T, \xi) - (q(\xi) - 2\alpha)g((AN)^T, (AN)^T) \\
 &= -\alpha(1 - \beta^2) + q(\xi) - \alpha - \alpha\beta^2 - (q(\xi) - 2\alpha)(1 - \beta^2) \\
 &= (q(\xi) - 2\alpha)\beta^2,
 \end{aligned}$$

where we have used  $g(A\phi S(AN)^T, \xi) = g(S(AN)^T, (AN)^T) = -\alpha(1 - \beta^2)$ . Here we note that  $\xi\beta = 0$ , because we can calculate the following

$$\begin{aligned}
 \xi\beta &= \xi g(A\xi, \xi) \\
 &= g((\bar{\nabla}_\xi A)\xi + A\bar{\nabla}_\xi \xi, \xi) + g(A\xi, \bar{\nabla}_\xi \xi) \\
 &= g(q(\xi)JA\xi, \xi) \\
 &= -q(\xi)g(A\xi, N) \\
 &= 0.
 \end{aligned}$$

Now we consider an open subset  $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$  in  $M$ . Then by (4.10), we have

**Lemma 4.1.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then*

$$q(\xi) = 2\alpha$$

*holds on  $\mathcal{U}$  on  $M$  in  $Q^{m*}$ .*

Now hereafter unless otherwise stated, on such an open subset  $\mathcal{U}$  let us prove that the unit vector field  $N$  in the complex hyperbolic quadric  $Q^{m*}$  is  $\mathfrak{A}$ -principal. Then by Lemma 4.1 and (4.4), we have the following for any tangent vector field  $X$  on  $M$

$$g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) = 0.$$

From this, by putting  $X = A\xi$  and using  $g(A\xi, A\xi) = 1$ , we know that

$$(4.11) \quad 0 = g(A\phi S A\xi, \xi) = g(S A\xi, (AN)^T).$$

Moreover, for any  $X \perp A\xi$  the second term in the left side of the above equation becomes

$$g(A\nabla_\xi X, \xi) = -g(X, \nabla_\xi A\xi) = \alpha g((AN)^T, X),$$

where in the third equality we have used Lemma 4.1. Consequently, for any tangent vector field  $X \perp A\xi$  we conclude

$$\begin{aligned} 0 &= g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) \\ &= g(X, S(AN)^T) + \alpha g((AN)^T, X) \\ &= g(S(AN)^T + \alpha(AN)^T, X). \end{aligned}$$

Moreover, by (4.11) we also know that

$$g(S(AN)^T + \alpha(AN)^T, A\xi) = 0.$$

So these two equations give the following

**Lemma 4.2.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then*

$$S(AN)^T = -\alpha(AN)^T$$

*holds on  $\mathcal{U}$  on  $M$  in  $Q^{m*}$ .*

Now let us differentiate the equation in Lemma 4.2. Then it follows that

$$(\nabla_X S)(AN)^T + S \nabla_X (AN)^T = -(X\alpha)(AN)^T - \alpha \nabla_X (AN)^T.$$

From this, by taking the inner product with the Reeb vector field  $\xi$  and using the formulas (3.3), we have

$$\begin{aligned} 0 &= g((AN)^T, (\nabla_X S)\xi) \\ &\quad + 2\alpha g(q(X)JAN - ASX - g(q(X)JAN - ASX, N)N, \xi) \\ &\quad + 2\alpha g(AN, N)g(SX, \xi) \\ &= g((AN)^T, \alpha\phi SX - S\phi SX) \\ &\quad + 2\alpha\{q(X)g(A\xi, \xi) - g(SX, A\xi) + g(AN, N)g(SX, \xi)\}. \end{aligned}$$

Then by putting  $X = (AN)^T$  and using Lemma 4.2, we have  $\alpha q((AN)^T) = 0$ . When the function  $\alpha = 0$ , in section 3,  $\beta g(Y, AN) = 0$  for any tangent vector field  $Y$  on  $M$ . Then on the open subset  $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$  in  $M$  we conclude

**Lemma 4.3.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then either*

$$q((AN)^T) = 0$$

*or the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal.*

On the other hand, by putting  $X = \xi$  in (3.3) and using Lemma 4.1, we have

$$(4.12) \quad \begin{aligned} \nabla_{\xi}(AN)^T &= (q(\xi) - \alpha)A\xi + \alpha g(AN, N)\xi \\ &= \alpha(A\xi - \beta\xi). \end{aligned}$$

Differentiating the equation in Lemma 4.2 along the Reeb direction  $\xi$  and using (4.12) implies

$$(4.13) \quad \begin{aligned} (\nabla_{\xi}S)(AN)^T &= -S\nabla_{\xi}(AN)^T - (\xi\alpha)(AN)^T - \alpha\nabla_{\xi}(AN)^T \\ &= -\alpha(SA\xi - \alpha\beta\xi) - (\xi\alpha)(AN)^T - \alpha^2(A\xi - \beta\xi). \end{aligned}$$

Moreover, differentiating  $S\xi = \alpha\xi$  and using Lemma 4.2, we get the following

$$(4.14) \quad \begin{aligned} (\nabla_{(AN)^T}S)\xi &= \{(AN)^T\alpha\}\xi + \alpha\phi S(AN)^T - S\phi S(AN)^T \\ &= \{(AN)^T\alpha\}\xi - \alpha^2\phi(AN)^T + \alpha S\phi(AN)^T. \end{aligned}$$

Then subtracting (4.14) from (4.13) and Lemma 4.2 give

$$(4.15) \quad \begin{aligned} g((\nabla_{\xi}S)(AN)^T - (\nabla_{(AN)^T}S)\xi, (AN)^T) &= -(\xi\alpha)(1 - \beta^2) \\ &= -g(\phi(AN)^T, (AN)^T) - g(\xi, A\xi)g(JA(AN)^T, (AN)^T) \\ &= 0, \end{aligned}$$

where in the second equality we have used the equation of Codazzi (3.1) in section 3. Then it follows that

$$\xi\alpha = 0 \quad \text{or} \quad \beta^2 = 1.$$

When the latter part  $\beta = \pm 1$  occurs on  $\mathcal{U}$ , then  $AN = \pm N$ . So we know that the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal. When  $\xi\alpha = 0$ , if we use the derivative formula  $Y\alpha$  and  $g(\xi, AN) = 0$  in section 3, we have the following

**Lemma 4.4.** *Let  $M$  be a Hopf real hypersurface in the complex hyperbolic quadric  $Q^{m*}$ ,  $m \geq 3$ . Then either*

$$\text{grad } \alpha = 2\beta(AN)^T$$

*or the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal.*

Now let us consider the first formula in Lemma 4.4. Then by differentiating the above formula it follows that

$$(4.16) \quad \begin{aligned} \nabla_X \text{grad } \alpha &= 2(X\beta)(AN)^T + 2\beta\nabla_X(AN)^T \\ &= 4g(A\phi S X, \xi)(AN)^T + 2\beta\{q(X)JAN - AS X \\ &\quad - g(q(X)JAN - AS X, N)N + g(AN, N)S X\}. \end{aligned}$$

Then we have

$$(4.17) \quad \begin{aligned} g(\nabla_X \text{grad } \alpha, Y) &= 4g(A\phi S X, \xi)g((AN)^T, Y) + 2\beta\{q(X)g(JAN, Y) - g(AS X, Y)\} \\ &\quad + 2\beta g(AN, N)g(S X, Y). \end{aligned}$$

Since  $g(\nabla_X \text{grad } \alpha, Y) = g(\nabla_Y \text{grad } \alpha, X)$  and Lemma 4.2, we have

$$(4.18) \quad 0 = 2\beta\{q(X)g(JAN, Y) - q(Y)g(JAN, X)\} - 2\beta\{g(ASX, Y) - g(ASY, X)\}.$$

So on the open subset  $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$  in  $M$  it follows that

$$q(X)g(JAN, Y) - q(Y)g(JAN, X) = g(ASX, Y) - g(ASY, X).$$

From this, by putting  $X = \xi$ , we know that

$$SA\xi = -\alpha A\xi + \beta \text{grad } q.$$

Then differentiating this formula gives

$$(4.19) \quad (\nabla_X S)A\xi + S\nabla_X A\xi = -(X\alpha)A\xi - \alpha\nabla_X A\xi + (X\beta)\text{grad } q + \beta\nabla_X \text{grad } q.$$

First let us take the inner product of (4.19) with  $Y$  and make the skew-symmetric part with respect  $X$  and  $Y$ . Next we use  $g(\nabla_X \text{grad } q, Y) = g(\nabla_Y \text{grad } q, X)$  to the obtained equation. Then finally by putting  $X = \xi$ , we have

$$(4.20) \quad \begin{aligned} &g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + g(S(\nabla_\xi A\xi), Y) - g(S(\nabla_Y A\xi), \xi) \\ &= -(\xi\alpha)g(A\xi, Y) + (Y\alpha)g(A\xi, \xi) \\ &\quad - \alpha\{g(\nabla_\xi A\xi, Y) - g(\nabla_Y A\xi, \xi)\} + (\xi\beta)q(Y) - (Y\beta)q(\xi). \end{aligned}$$

In this equation (4.20), we want to use the following formulas

$$q(\xi) = 2\alpha, \quad \xi\alpha = 0, \quad \xi\beta = 0,$$

$$(4.21) \quad \begin{aligned} \nabla_\xi(A\xi) &= 2\alpha JA\xi + \alpha AN - \{2\alpha g(JA\xi, N) + \alpha g(AN, N)\}N \\ &= -\alpha AN - \alpha\beta N \\ &= -\alpha(AN)^T, \end{aligned}$$

and

$$(4.22) \quad \begin{aligned} g(\nabla_Y(A\xi), \xi) &= q(Y)g(JA\xi, \xi) + g(A\phi SY, \xi) \\ &= g(SY, AN) = -\alpha g((AN)^T, Y). \end{aligned}$$

Then by the help of (4.21) and (4.22), the equation (4.20) can be reformed as

$$(4.23) \quad \begin{aligned} &g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + 2\alpha^2 g((AN)^T, Y) \\ &= (Y\alpha)\beta - 2\alpha(Y\beta). \end{aligned}$$

On the other hand, if we use the equation of Codazzi (3.1) in the first term of (4.23), we have

$$(4.24) \quad \begin{aligned} g((\nabla_\xi S)A\xi, Y) &= g((\nabla_\xi S)Y, A\xi) = g((\nabla_Y S)\xi, A\xi) \\ &\quad - g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi). \end{aligned}$$

Then substituting (4.24) into the first term of (4.23) gives

$$(4.25) \quad \begin{aligned} &-g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi) + 2\alpha^2 g((AN)^T, Y) \\ &= (Y\alpha)\beta - 2\alpha(Y\beta) \end{aligned}$$

$$= 2\beta^2 g(Y, AN) + 4\alpha^2 g(Y, (AN)^T),$$

where in the second equality we have used  $\xi\alpha = 0$  in (3.2) of section 3, Lemma 4.2 and (3.8) in the following formula

$$\begin{aligned} Y\beta &= 2g(A\phi S Y, \xi) = 2g(S Y, AJ\xi) \\ &= 2g(S Y, (AN)^T) = -2\alpha g(Y, (AN)^T). \end{aligned}$$

In (4.25) the first two terms of the left side cancelled out each other and the third term vanishes identically. The fourth term  $2\alpha^2 g((AN)^T, Y)$  can be deleted with the second term in the right side of (4.25). So (4.25) implies  $2(\alpha^2 + \beta^2)g(Y, AN) = 0$  for any tangent vector field  $Y$  on  $M$ , which means that on the open subset  $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$  the unit normal vector field  $N$  is  $\mathfrak{A}$ -principal  $AN = g(AN, N)N$ .

Summing up the above discussions, we can prove our Main Theorem 1 in the introduction.

By virtue of Main Theorem 1, we can distinguish two classes of real hypersurfaces in the complex hyperbolic quadric  $Q^{m*}$  with Killing structure Jacobi operator : those that have  $\mathfrak{A}$ -principal unit normal, and those that have  $\mathfrak{A}$ -isotropic unit normal vector field  $N$ . We treat the respective cases in sections 5 and 6.

### 5. Killing structure Jacobi operator with $\mathfrak{A}$ -principal normal

In this section we consider a real hypersurface  $M$  in the complex hyperbolic quadric  $Q^{m*}$  with  $\mathfrak{A}$ -principal unit normal vector field. Then the unit normal vector field  $N$  satisfies  $AN = N$  for a complex conjugation  $A \in \mathfrak{A}$ . Naturally, we have also the following

$$A\xi = -\xi, \quad \text{and} \quad JA\xi = -J\xi = -N.$$

Then the structure Jacobi operator  $R_\xi$  is given by

$$(5.1) \quad R_\xi(X) = -X + 2\eta(X)\xi + AX + g(S\xi, \xi)SX - g(SX, \xi)S\xi.$$

Since we assume that  $M$  is Hopf, (5.1) becomes

$$(5.2) \quad R_\xi(X) = -X + 2\eta(X)\xi + AX + \alpha SX - \alpha^2 \eta(X)\xi.$$

By the assumption of the Killing structure Jacobi operator  $R_\xi$ , the derivative of  $R_\xi$  along any tangent vector field  $Y$  on  $M$  is given by

$$\begin{aligned} (5.3) \quad (\nabla_Y R_\xi)(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= 2\{(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi\} + (\nabla_Y A)X + (Y\alpha)SX \\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y \eta)(X)\xi - \alpha^2 \eta(X)\nabla_Y \xi. \end{aligned}$$

We can write

$$AY = BY + \rho(Y)N,$$

where  $BY$  denotes the tangential component of  $AY$  and  $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$ . So for any tangent vector field  $Y$  on  $M$  the vector field  $AY (= BY)$  also becomes

a tangent vector field on  $M$  in  $Q^{m*}$ . Then it follows

$$\begin{aligned}
 (5.4) \quad (\nabla_Y A)X &= \nabla_Y(AX) - A\nabla_Y X \\
 &= \bar{\nabla}_Y(AX) - \sigma(Y, AX) - A\nabla_Y X \\
 &= (\bar{\nabla}_Y A)X + A\bar{\nabla}_Y X - \sigma(Y, AX) - A\nabla_Y X \\
 &= q(Y)JAX + A\sigma(Y, X) - \sigma(Y, AX) \\
 &= q(Y)JAX + g(SX, Y)AN - g(SY, AX)N,
 \end{aligned}$$

where we have used the equation of Gauss in the second equality and the Weingarten formula in the fifth equality. From this, together with (5.3) and using that  $\mathfrak{A}$ -principal, the Killing structure Jacobi operator gives

$$\begin{aligned}
 (5.5) \quad 0 &= (\nabla_Y R_\xi)(X) + (\nabla_X R_\xi)(Y) \\
 &= (2 + \alpha^2)\{(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi\} \\
 &\quad + (2 + \alpha^2)\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} \\
 &\quad + \{q(Y)JAX + g(SX, Y)N - g(SY, AX)N\} \\
 &\quad + \{q(X)JAY + g(SY, X)N - g(SX, AY)N\} \\
 &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
 &\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi.
 \end{aligned}$$

From this, taking the inner product of (5.5) with the Reeb vector field  $\xi$ , and using the constancy of the Reeb function  $\alpha$  in Lemma 3.2, we have

$$\begin{aligned}
 (5.6) \quad 0 &= (2 + \alpha^2)\{g(\phi SY, X) + g(\phi SX, Y)\} + \alpha g((\nabla_Y S)X + (\nabla_X S)Y, \xi) \\
 &= 2g((\phi S - S\phi)Y, X)
 \end{aligned}$$

where we have used  $g(JAX, \xi) = -g(AX, N) = -g(X, AN) = -g(X, N) = 0$  for any tangent vector field  $X$  on  $M$  in  $Q^{m*}$  and  $(\nabla_X S)\xi = \alpha\phi SX - S\phi SX$ . The formula (5.6) means that the shape operator  $S$  commutes with the structure tensor  $\phi$ . Then by Theorem A in the introduction,  $M$  is locally congruent to an open part of a tube around a totally geodesic  $\mathbb{C}H^k \subset Q^{2k*}$  or a horosphere whose center at infinity is  $\mathfrak{A}$ -isotropic singular. That is, the Reeb flow on  $M$  is isometric.

On the other hand, we want to introduce the following proposition (see Suh [34]).

**Proposition 5.1.** *Let  $M$  be a real hypersurface in  $Q^{m*}$ ,  $m \geq 3$ , with isometric Reeb flow. Then the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic everywhere.*

By Proposition 5.1, we know that the unit normal vector field  $N$  of  $M$  is  $\mathfrak{A}$ -isotropic, not  $\mathfrak{A}$ -principal. This rules out the existence of an  $\mathfrak{A}$ -principal unit normal  $N$  together with Killing structure Jacobi operator. So we give the proof of our Main Theorem 2 with  $\mathfrak{A}$ -principal unit normal  $N$ .

## 6. Killing structure Jacobi operator with $\mathfrak{A}$ -isotropic normal

In this section we assume that the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic and the Reeb

function  $\alpha = g(S\xi, \xi)$  is constant along the Reeb direction  $\xi$ , that is,  $\xi\alpha = 0$ . Then the normal vector field  $N$  can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for  $Z_1, Z_2 \in V(A)$ , where  $V(A)$  denotes a  $(+1)$ -eigenspace of the complex conjugation  $A \in \mathfrak{A}$ . Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0.$$

By virtue of these formulas for  $\mathfrak{A}$ -isotropic unit normal, the structure Jacobi operator can be given as follows:

$$\begin{aligned} (6.1) \quad R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi + g(AX, \xi)A\xi + g(JAX, \xi)JA\xi \\ &\quad + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned}$$

On the other hand, we know that  $JA\xi = -JAJN = AJ^2N = -AN$ , and  $g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N)$ . Then the structure Jacobi operator  $R_\xi$  can be rearranged as follows:

$$\begin{aligned} (6.2) \quad R_\xi(X) &= -X + \eta(X)\xi + g(AX, \xi)A\xi + g(X, AN)AN \\ &\quad + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned}$$

Then by differentiating (6.2), we obtain

$$\begin{aligned} (6.3) \quad \nabla_Y R_\xi(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= (\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi + g(X, \nabla_Y(A\xi))A\xi \\ &\quad + g(X, A\xi)\nabla_Y(A\xi) + g(X, \nabla_Y(AN))AN + g(X, AN)\nabla_Y(AN) \\ &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y \eta)(X)\xi - \alpha^2\eta(X)\nabla_Y \xi. \end{aligned}$$

Here let us consider the equation of Gauss. It is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

for any vector fields  $X$  and  $Y$  on  $M$  in  $Q^{m*}$ , where  $\nabla_X Y = (\bar{\nabla}_X Y)^T$  and  $\sigma(X, Y)$  respectively denote the tangential and normal component on  $T_z M$  of  $\bar{\nabla}_X Y$  in  $T_z Q^{m*}$ ,  $z \in M$ . The Weingarten formula is given by

$$\bar{\nabla}_X N = -SX$$

for an  $\mathfrak{A}$ -isotropic unit normal vector field  $N$ . Here  $S$  denotes the shape operator of  $M$  in the complex hyperbolic quadric  $Q^{m*}$  derived from the unit normal  $N$ . Then by using these two equations to some terms in (6.3), we have the following :



$$\begin{aligned}
\nabla_Y(A\xi) &= \bar{\nabla}_Y(A\xi) - \sigma(Y, A\xi) \\
&= (\bar{\nabla}_Y A)\xi + A\bar{\nabla}_Y \xi - \sigma(Y, A\xi) \\
&= q(Y)JA\xi + A\{\phi S Y + \eta(S Y)N\} - g(S Y, A\xi)N
\end{aligned}$$

and

$$\begin{aligned}
\nabla_Y(AN) &= \bar{\nabla}_Y(AN) - \sigma(Y, AN) \\
&= (\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N - \sigma(Y, AN) \\
&= q(Y)JAN - AS Y - g(S Y, AN)N.
\end{aligned}$$

Substituting these formulas into (6.3) and using the assumption of Killing structure Jacobi operator, we have

$$\begin{aligned}
(6.4) \quad 0 &= \nabla_Y R_\xi(X) + \nabla_X R_\xi(Y) \\
&= g(\phi S Y, X)\xi + \eta(X)\phi S Y \\
&\quad + g(\phi S X, Y)\xi + \eta(Y)\phi S X \\
&\quad + \{q(Y)g(A\xi, X) + g(A\phi S Y, X) + g(S Y, \xi)g(AN, X)\}A\xi \\
&\quad + \{q(X)g(A\xi, Y) + g(A\phi S X, Y) + g(S X, \xi)g(AN, Y)\}A\xi \\
&\quad + g(X, A\xi)\{q(Y)JA\xi + A\phi S Y + g(S Y, \xi)AN - g(S Y, A\xi)N\} \\
&\quad + g(Y, A\xi)\{q(X)JA\xi + A\phi S X + g(S X, \xi)AN - g(S X, A\xi)N\} \\
&\quad + \{q(Y)g(X, AN) - g(X, AS Y)\}AN \\
&\quad + \{q(X)g(Y, AN) - g(Y, AS X)\}AN \\
&\quad + g(X, AN)\{q(Y)JAN - AS Y - g(S Y, AN)N\} \\
&\quad + g(Y, AN)\{q(X)JAN - AS X - g(S X, AN)N\} \\
&\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
&\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi \\
&\quad - \alpha^2 g(\phi S Y, X)\xi - \alpha^2 \eta(X)\phi S Y \\
&\quad - \alpha^2 g(\phi S X, Y)\xi - \alpha^2 \eta(Y)\phi S X.
\end{aligned}$$

Taking the inner product of (6.4) with the unit normal  $N$  and using the properties of  $\mathfrak{A}$ -isotropic, that is,  $g(A\xi, \xi) = 0$ ,  $g(AN, N) = 0$ , it follows that

$$\begin{aligned}
(6.5) \quad 0 &= g(X, A\xi)g(A\phi S Y, N) - g(X, A\xi)g(S Y, A\xi) \\
&\quad + g(Y, A\xi)g(A\phi S X, N) - g(Y, A\xi)g(S X, A\xi) \\
&\quad - g(X, AN)g(AS Y, N) - g(X, AN)g(S Y, AN) \\
&\quad - g(Y, AN)g(AS X, N) - g(Y, AN)g(S X, AN).
\end{aligned}$$

From this, putting  $X = AN$  and using that  $N$  is  $\mathfrak{A}$ -isotropic and  $A\xi = \phi AN$ , we have

$$0 = -2g(AS Y, N) - 2g(Y, AN)g(S AN, AN) + 2g(Y, A\xi)g(A\phi S AN, N).$$

By putting  $Y = AN$ , we get  $g(S AN, AN) = 0$ . Then the above equation reduces to

$$g(AS Y, N) = g(Y, A\xi)g(A\phi S AN, N).$$

So it follows that

$$\begin{aligned} SAN &= g(A\phi SAN, N)A\xi \\ &= -g(SAN, \phi AN)A\xi \\ &= -g(SAN, A\xi)A\xi, \end{aligned}$$

where we have used  $A\xi = \phi AN$ . Then this gives that  $g(SAN, A\xi) = 0$ , which implies

$$(6.6) \quad SAN = 0 \quad \text{and} \quad S\phi A\xi = 0.$$

Then (6.5) reduces to the following

$$(6.7) \quad \begin{aligned} 0 &= g(X, A\xi)g(A\phi SY, N) - g(X, A\xi)g(SY, A\xi) \\ &\quad + g(Y, A\xi)g(A\phi SX, N) - g(Y, A\xi)g(SX, A\xi). \end{aligned}$$

By putting  $X = A\xi$  in (6.7) and using  $A\xi = \phi AN$ , it follows that

$$g(SY, A\xi) + g(Y, A\xi)g(SA\xi, A\xi) = 0$$

for any vector field  $Y$  on  $M$  in  $Q^{m*}$ . This gives

$$SA\xi = -g(SA\xi, A\xi)A\xi.$$

Then by taking the inner product with  $A\xi$ , we know  $g(SA\xi, A\xi) = 0$ . From this, together with the above equation, we have

$$(6.8) \quad SA\xi = 0 \quad \text{and} \quad S\phi AN = 0.$$

Putting  $X = \xi$  into (6.4), and using (6.8) and the  $\mathfrak{U}$ -isotropic property  $g(A\xi, \xi) = 0$ , we have

$$(6.9) \quad \begin{aligned} 0 &= \phi SY + \{q(\xi)g(A\xi, Y) + \alpha g(AN, Y)\}A\xi \\ &\quad + g(Y, A\xi)\{q(\xi)A\xi + \alpha AN - g(S\xi, A\xi)N\} \\ &\quad + \{q(\xi)g(Y, AN) - \alpha g(Y, A\xi)\}AN + g(Y, AN)\{q(\xi)AN - \alpha A\xi\} \\ &\quad + (Y\alpha)\alpha\xi + \alpha(\nabla_Y S)\xi - (Y\alpha^2)\xi - \alpha^2\phi SY \\ &\quad + (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - (\xi\alpha^2)\eta(Y)\xi \\ &= \phi SY + 2q(\xi)g(A\xi, Y)A\xi + 2q(\xi)g(Y, AN)AN \\ &\quad - \alpha S\phi SY + (\xi\alpha)SY - (\xi\alpha^2)\eta(Y)\xi + \alpha(\nabla_\xi S)Y. \end{aligned}$$

On the other hand,  $SA\xi = 0$  implies  $(\nabla_\xi S)A\xi + S\nabla_\xi(A\xi) = 0$ . By the equation of Gauss, the following holds

$$\begin{aligned} \nabla_\xi(A\xi) &= \bar{\nabla}_\xi(A\xi) - \sigma(\xi, A\xi) \\ &= q(\xi)JA\xi + g(S\xi, \xi)AN - g(S\xi, A\xi)N \\ &= q(\xi)JA\xi + \alpha AN. \end{aligned}$$

This gives  $S(\nabla_\xi(A\xi)) = q(\xi)SJA\xi + \alpha SAN = 0$  from (6.6). From this, together with the above formula, we have

$$(6.10) \quad (\nabla_\xi S)A\xi = 0.$$

By taking the inner product of (6.9) with  $A\xi$  and  $AN$  respectively, and using (6.6), (6.8)

and (6.10), we know that

$$q(\xi)A\xi = 0 \quad \text{and} \quad q(\xi)AN = 0.$$

By virtue of these formulas, (6.9) reduces to the following

$$(6.11) \quad 0 = \phi S Y - \alpha S \phi S Y + (\xi \alpha) S Y - (\xi \alpha^2) \eta(Y) \xi + \alpha (\nabla_\xi S) Y.$$

On the other hand, by using the equation of Codazzi, we have

$$\begin{aligned} (\nabla_\xi S) Y &= (\nabla_Y S) \xi - \phi Y + g(AN, Y) A \xi + g(Y, A \xi) \phi A \xi \\ &= (Y \alpha) \xi + \alpha \phi S Y - S \phi S Y - \phi Y \\ &\quad + g(AN, Y) A \xi + g(Y, A \xi) \phi A \xi. \end{aligned}$$

Then by the properties of  $M$  being Hopf and with  $\mathfrak{U}$ -isotropic unit normal vector field, we have

$$Y \alpha = g((\nabla_\xi S) Y, \xi) = g((\nabla_\xi S) \xi, Y) = (\xi \alpha) \eta(Y).$$

From this, together with the assumption of  $\xi \alpha = 0$  in section 6, it follows that the Reeb function  $\alpha$  is constant for a real hypersurface in  $Q^{m*}$  with  $\mathfrak{U}$ -isotropic unit normal. Then the derivative of the shape operator  $S$  along the Reeb direction  $\xi$  is given by

$$\begin{aligned} -\alpha (\nabla_\xi S) Y &= -\alpha^2 \phi S Y + \alpha S \phi S Y \\ &\quad + \alpha \phi Y - \alpha g(AN, Y) A \xi - \alpha g(Y, A \xi) \phi A \xi. \end{aligned}$$

From this, by (6.11) and using the constancy of the Reeb function  $\alpha$ , we know that

$$(6.12) \quad \begin{aligned} 0 &= \phi S Y - 2\alpha S \phi S Y + \alpha^2 \phi S Y \\ &\quad - \alpha \phi Y + \alpha g(AN, Y) A \xi + \alpha g(Y, A \xi) \phi A \xi. \end{aligned}$$

Then for any  $Y \in \mathcal{Q}$  such that  $S Y = \lambda Y$ , where  $Y$  is orthogonal to the vector fields  $A \xi$  and  $AN$ , (6.12) reduces to the following

$$(6.13) \quad 2\alpha \lambda S \phi Y = (\lambda \alpha^2 - \alpha + \lambda) \phi Y.$$

Then (6.13) gives  $\alpha \neq 0$ .

In fact, if the Reeb function  $\alpha = 0$ , from (6.13) it follows that  $\lambda = 0$ . From this, together with (6.6) and (6.8), the shape operator  $S$  becomes identically vanishing. That is,  $M$  is totally geodesic. Then by the equation of Codazzi in section 3, we have a contradiction.

Naturally we should have  $2\alpha \lambda \neq 0$ . If the function  $\lambda = 0$ , then (6.13) also implies that the Reeb function  $\alpha$  vanishes. So also the contradiction appears. This fact gives

$$S \phi Y = \frac{\alpha \lambda - 2}{2\lambda - \alpha} \phi Y = \frac{\alpha^2 \lambda - \alpha + \lambda}{2\alpha \lambda} \phi Y.$$

It can be written as follows:

$$(6.14) \quad 2\lambda^2 + \alpha(1 - \alpha^2)\lambda + \alpha^2 = 0.$$

Then the discriminant of (6.14) is given by

$$D = \alpha^2(1 - \alpha^2)^2 - 8\alpha^2 = \alpha^2\{(\alpha^2 - 1)^2 - 8\}.$$

Then the solution has two roots as follows:

$$\lambda = \frac{-\alpha(1 - \alpha^2) \pm \alpha \sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}.$$

When  $\alpha^2 > 2\sqrt{2} + 1$ , we have two distinct roots  $\lambda_1$  and  $\lambda_2$  of the equation (6.14).

Now let us consider the case that  $\alpha^2 = 2\sqrt{2} + 1$ . Then we may put  $\alpha = \sqrt{2\sqrt{2} + 1}$ . In this case we have

$$\lambda_1 = \lambda_2 = \frac{-\alpha(1 - \alpha^2)}{4} = -\sqrt{\sqrt{2} + \frac{1}{2}}.$$

Here let us put  $\delta = -\sqrt{\sqrt{2} + \frac{1}{2}}$ . Then the shape operator  $S$  has three distinct constant principal curvatures such that

$$\alpha = \sqrt{2\sqrt{2} + 1}, \quad \beta = \gamma = 0, \quad \text{and} \quad \delta = -\sqrt{\sqrt{2} + \frac{1}{2}} = -\sqrt{\frac{2\sqrt{2} + 1}{2}}.$$

The corresponding eigen spaces are given by  $\xi \in T_0$ ,  $A\xi, AN \in T_\beta = \mathcal{Q}^\perp$  and  $T_\delta = \mathcal{Q}$  with multiplicities 1, 2 and  $2m - 4$  respectively.

On the other hand, on the distribution  $\mathcal{Q}$  let us introduce an important formula mentioned in section 3 as follows:

$$(6.15) \quad 2S\phi SY - \alpha(\phi S + S\phi)Y = -2\phi Y$$

for any tangent vector field  $Y$  on  $M$  in  $\mathcal{Q}^m$  (see also [29], pages 1350050-11). So if  $SY = \delta Y$  in (6.15), then  $(2\delta - \alpha)S\phi Y = (\alpha\delta - 2)\phi Y$ , which gives

$$(6.16) \quad S\phi Y = \frac{\alpha\delta - 2}{2\delta - \alpha}\phi Y,$$

because if  $2\delta - \alpha = 0$ , then  $\alpha\delta - 2 = 0$ . This implies  $\alpha^2 = 4$ , then  $\alpha = 2$  and  $\delta = 1$ . In this case  $M$  is locally congruent to a horosphere whose center at infinity is  $\mathfrak{U}$ -isotropic singular.

On the other hand, let us consider  $S\phi Y = \delta\phi Y$  for  $2\delta \neq \alpha$ , because  $T_\delta = \mathcal{Q}$ . From this, together with the above equation, we have

$$\delta^2 - \alpha\delta + 1 = 0.$$

Then  $\delta^2 + 1 = \sqrt{2} + \frac{3}{2}$ . But  $\delta^2 + 1 = \alpha\delta = -\sqrt{2\sqrt{2} + 1}\sqrt{\frac{2\sqrt{2} + 1}{2}} = -\frac{\sqrt{2}}{2} - 2$ . This gives a contradiction. So this case can not be happened.

Accordingly, the shape operator  $S$  can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_2 \end{bmatrix}$$

where the principal curvatures are constants and are given by

$$\lambda_1 = \frac{\alpha(\alpha^2 - 1) + \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}$$

and respectively

$$\lambda_2 = \frac{\alpha(\alpha^2 - 1) - \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}.$$

By virtue of Remark below, we note that the horosphere whose center at infinity is  $\mathfrak{U}$ -isotropic singular can not be appeared. Then we give a complete proof of our Main Theorem 3.

REMARK 6.1. Let us check that a tube around the totally geodesic  $\mathbb{C}H^k \subset Q^{2k*}$  or a horosphere whose center at infinity is  $\mathfrak{U}$ -isotropic singular. Then by Theorem A in the introduction, the tube has a commuting shape operator, that is,  $S\phi = \phi S$  and the unit normal  $N$  is  $\mathfrak{U}$ -isotropic and the Reeb curvature  $\alpha = g(S\xi, \xi)$  is constant (see Suh [34]). By the  $\mathfrak{U}$ -isotropic unit normal, the properties  $g(A\xi, \xi) = 0$  and  $g(AN, N) = 0$  hold on  $M$ . Moreover from the expression of this tube we know that  $SA\xi = 0$  and  $SAN = 0$ , by differentiating we also confirm that  $(\nabla_\xi S)A\xi = 0$  and  $(\nabla_\xi S)AN = 0$ .

Now we assume that the tube admits a Killing structure Jacobi operator. Then by the same process as in the proof of our Main Theorem 2, the principal curvature of the tube should satisfies (6.14), that is,

$$2\lambda^2 + \alpha(1 - \alpha^2)\lambda + \alpha^2 = 0.$$

Then two roots  $\coth r$  and  $\tanh r$  of the tube should satisfy  $1 = \lambda\mu = \coth r \cdot \tanh r = \frac{\alpha^2}{2}$ . Then  $2 = \alpha^2 = \coth^2 r + \tanh^2 r + 2$  implies  $\coth^2 r + \tanh^2 r = 0$ . This makes a contradiction. So the tube does not admit a Killing structure Jacobi operator. Then naturally the tube around the totally geodesic  $\mathbb{C}H^k \subset Q^{2k*}$  or the horosphere does not have a parallel structure Jacobi operator, which is more strong condition than Killing structure Jacobi operator.

ACKNOWLEDGEMENTS. The present author would like to express his hearty thanks to the referee for his/her valuable comments and suggestions to improve the first version of our manuscript.

### References

- [1] A.L. Besse: *Einstein Manifolds*, Springer-Verlag, Berlin, 2008.
- [2] D.E. Blair: *Almost contact manifolds with Killing structure tensors*, Pacific J. Math. **39** (1971), 285–292.
- [3] S. Helgason: *Differential geometry, Lie groups and symmetric spaces*, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI, 2001.
- [4] I. Jeong, Y.J. Suh and C. Woo: *Real hypersurfaces in complex two-plane Grassmannian with recurrent structure Jacobi operator*; in Real and complex submanifolds, Springer Proc. in Math. & Statistics **106** (2014), 267–278.
- [5] U.-H. Ki, J.D. Pérez, F.G. Santos and Y.J. Suh: *Real hypersurfaces in complex space form with  $\xi$ -parallel Ricci tensor and structure Jacobi operator*, J. Korean Math. Soc. **44** (2007), 307–326.
- [6] M. Kimura: *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.
- [7] M. Kimura, I. Jeong, H. Lee and Y.J. Suh: *Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster Reeb parallel shape operator*, Monatsh. Math. **171** (2013), 357–376.
- [8] S. Klein: *Totally geodesic submanifolds of the complex quadric*, Differential Geom. Appl. **26** (2008), 79–96.
- [9] S. Klein: *Totally geodesic submanifolds of the complex and the quaternionic 2-Grassmannians*, Trans. Amer. Math. Soc. **361** (2009), 4927–4967.
- [10] S. Klein and Y.J. Suh: *Contact real hypersurfaces in the complex hyperbolic quadric*, Ann. Mat. Pura Appl. **198(4)** (2019), 1481–1494.
- [11] A.W. Knap: *Lie Groups Beyond an Introduction*, Progress in Mathematics **140**, Birkhäuser Boston, Inc., Boston, MA, 2002.
- [12] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry*, Vol. II, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996.
- [13] S. Montiel and A. Romero: *On some real hypersurfaces in a complex hyperbolic space*, Geom. Dedicata **20** (1986), 245–261.
- [14] S. Montiel and A. Romero: *Holomorphic sectional curvatures indefinite complex Grassmann manifolds*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 121–125.
- [15] S. Montiel and A. Romero: *Complex Einstein hypersurfaces of indefinite complex space forms*, Math. Proc. Cambridge Philos. Soc. **94** (1983), 495–508.
- [16] M. Okumura: *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.
- [17] J.D. Pérez: *Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space*, Ann. Mat. Pura Appl. **194** (2015), 1781–1794.
- [18] J.D. Pérez and F.G. Santos: *Real hypersurfaces in complex projective space with recurrent structure Jacobi operator*, Differential Geom. Appl. **26** (2008), 218–223.
- [19] J.D. Pérez, F.G. Santos and Y.J. Suh: *Real hypersurfaces in complex projective space whose structure Jacobi operator is Lie  $\xi$ -parallel*, Differential Geom. Appl. **22** (2005), 181–188.
- [20] J.D. Pérez, I. Jeong and Y.J. Suh: *Real hypersurfaces in complex two-plane Grassmannian with parallel structure Jacobi operator*, Acta. Math. Hungar. **22** (2009), 173–186.
- [21] J.D. Pérez, F.G. Santos and Y.J. Suh: *Real hypersurfaces in complex projective space whose structure Jacobi operator is  $D$ -parallel*, Bull. Belg. Math. Soc. Simon Stevin **13** (2006), 459–469.
- [22] H. Reckziegel: *On the geometry of the complex quadric*; in Geometry and Topology of Submanifolds VIII (Brussels/Nordfjordeid 1995), World Sci. Publ., River Edge, NJ, 1995, 302–315.
- [23] B. Smyth: *Differential geometry of complex hypersurfaces*, Ann. Math. **85** (1967), 246–266.
- [24] B. Smyth: *Homogeneous complex hypersurfaces*, J. Math. Soc. Japan **20** (1968), 643–647.
- [25] K. Nomizu: *On the rank and curvature of non-singular complex hypersurfaces in complex projective space*, J. Math. Soc. Japan **21** (1967), 266–269.
- [26] Y.J. Suh: *Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1309–1324.
- [27] Y.J. Suh: *Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature*, J. Math. Pures Appl. **100** (2013), 16–33.
- [28] Y.J. Suh: *Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians*, Adv. in Appl. Math. **50** (2013), 645–659.

- [29] Y.J. Suh: *Real hypersurfaces in the complex quadric with Reeb parallel shape operator*, Internat. J. Math. **25** (2014), 1450059, 17pp.
- [30] Y.J. Suh: *Real hypersurfaces in the complex quadric with Reeb invariant shape operator*, Differential Geom. Appl. **38** (2015), 10–21.
- [31] Y.J. Suh: *Real hypersurfaces in the complex quadric with parallel Ricci tensor*, Adv. Math. **281** (2015), 886–905.
- [32] Y.J. Suh: *Real hypersurfaces in the complex quadric with harmonic curvature*, J. Math. Pures Appl. **106** (2016), 393–410.
- [33] Y.J. Suh: *Real hypersurfaces in the complex quadric with parallel normal Jacobi operator*, Math. Nachr. **290** (2017), 442–451.
- [34] Y.J. Suh: *Real hypersurfaces in the complex hyperbolic quadrics with isometric Reeb flow*, Commun. Contemp. Math. **20** (2018), 1750031, 20pp.
- [35] Y.J. Suh: *Real hypersurfaces in the complex hyperbolic quadric with parallel normal Jacobi operator*, Mediterr. J. Math. **15** (2018), no. 159, 14pp.
- [36] Y.J. Suh and D.H. Hwang: *Real hypersurfaces in the complex hyperbolic quadric with Reeb parallel shape operator*, Ann. Mat. Pura Appl. **196** (2017), 1307–1326.
- [37] Y.J. Suh and C. Woo: *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor*, Math. Nachr. **287** (2014), 1524–1529.
- [38] Y.J. Suh, G. Kim and C. Woo: *Pseudo anti-commuting Ricci tensor and Ricci soliton real hypersurfaces in complex hyperbolic two-plane Grassmannians*, Math. Nachr. **291** (2018), 1574–1594.
- [39] Y.J. Suh, J.D. Pérez and C. Woo: *Real hypersurfaces in the complex hyperbolic quadric with parallel structure Jacobi operator*, Publ. Math. Debrecen **94** (2019), 75–107.
- [40] S. Tachibana: *On Killing tensors in a Riemannian space*, Tohoku Math. J. **20** (1968), 257–264.
- [41] R. Takagi: *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
- [42] K. Yano: *Some remarks on tensor fields and curvature*, Ann. of Math. **55** (1952), 328–347.

Kyungpook National University  
 College of Natural Sciences  
 Department of Mathematics  
 and Research Institute of Real &  
 Complex Manifolds  
 Daegu 41566  
 Republic of Korea  
 e-mail: yjsuh@knu.ac.kr